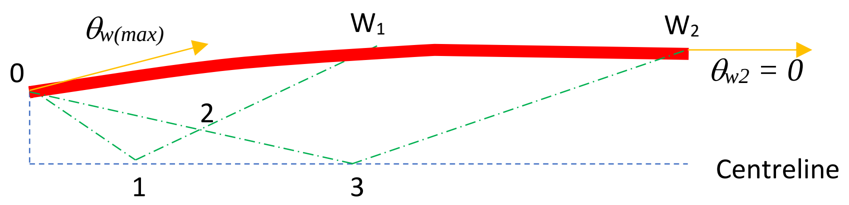
Worked Example of Method of Characteristics (minimum length nozzle)

This example is based on a Youtube video lecture entitled "lec54 Method of Characteristics-Applications" from "NPTEL - Indian Institute of Science, Bengaluru" given by Dr S Rao: [lec54 Method of Characteristics- Applications - YouTube]

Setup

Consider the nozzle shown below:



Setup and initial conditions:

- We'll say that the flow at the throat (and therefore point 0) is M = 1 and uniform (in reality though this "sonic line" wouldn't be straight - that is the parameters wouldn't be constant in a straight vertical line like this – leading to some inaccuracies).
- The angle of the wall (and therefore the flow) at point zero is the maximum angle of expansion and is given by:

$$\theta_{w(\max)} = \frac{\nu(M_e)}{2}$$

  As shown on the diagram. where $\nu(M_e)$ is the Prandtl-Meyer number of the flow at the exit (Mach 2 in this example).

- The flow along the centre line is parallel to it ($\theta = 0^o$)
- The x, y coordinates of point 0 are (0, 1).
- At the exit of the nozzle M = 2 (this is what we are designing for).
- The angle of the wall at W_2 that is $\theta_{w2} = 0$ because at this point, we want the wall of the nozzle to be parallel with the exit flow.
- The exit flow will be assumed to uniform.

Procedure

- Proceed from left to right calculating the values at each point.
- The angle of the wall points in the straightening section will be equal to the flow angle at that point (this cancels out the shockwaves at that point).

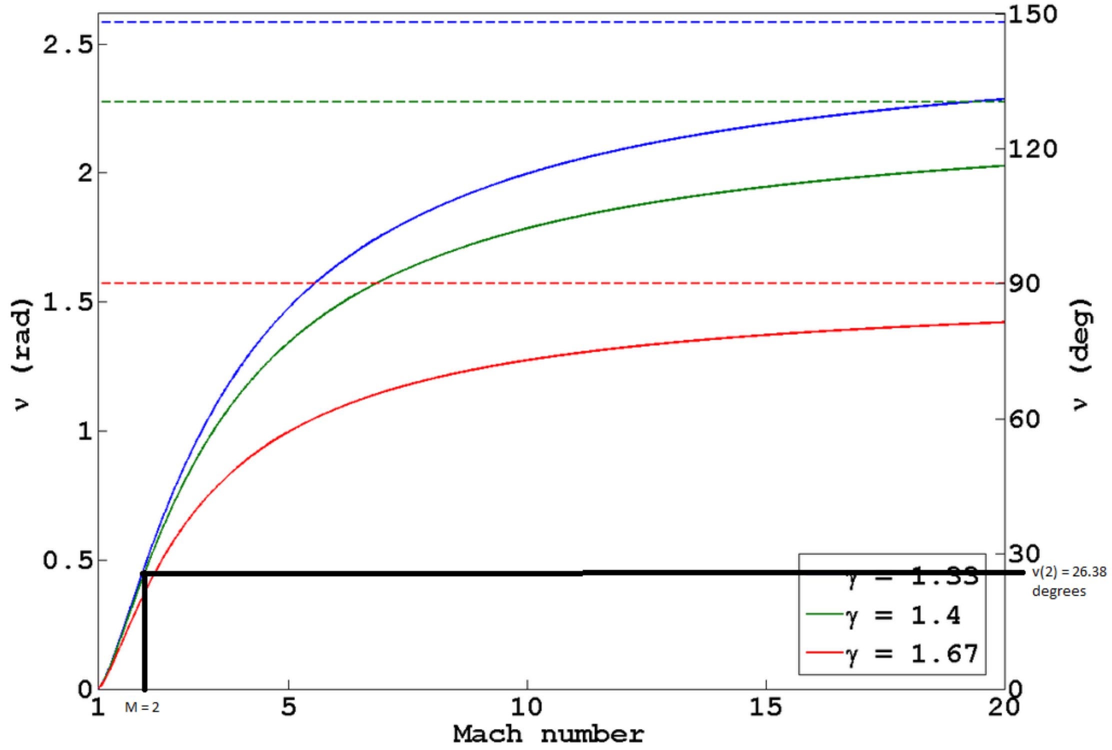
Calculations

At point 0 (throat):

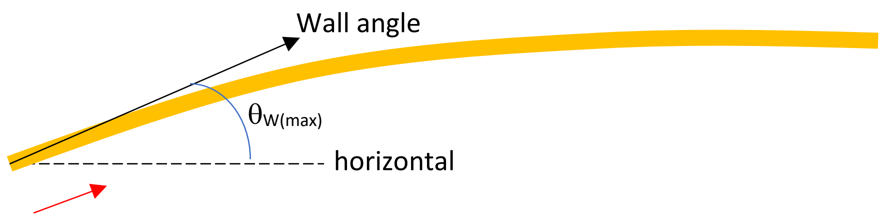
Conditions as stated above (M = 1) and:

$$\theta_{w(\max)} = \frac{\upsilon(M_e)}{2} = \frac{\upsilon(2)}{2} = \frac{26.38}{2} = 13.19^o$$

You are best to use an on-line calculator (eg: [Prandtl-Meyer Angle (nasa.gov)]) to calculate this – but the graph is shown below for reference:

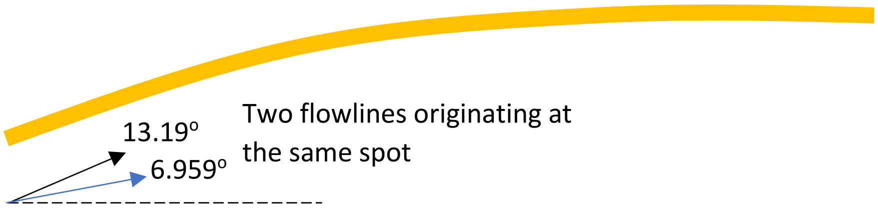


This is not only the angle of the wall at the sonic line – because the flow must follow the curve of the wall for isentropic conditions - it is also the angle of flow at the wall:



The characteristic line originating from this flow will eventually be the one terminating at the mouth of the nozzle (the one going from point 0 to point 3 and eventually point W_2) in the first-page diagram.

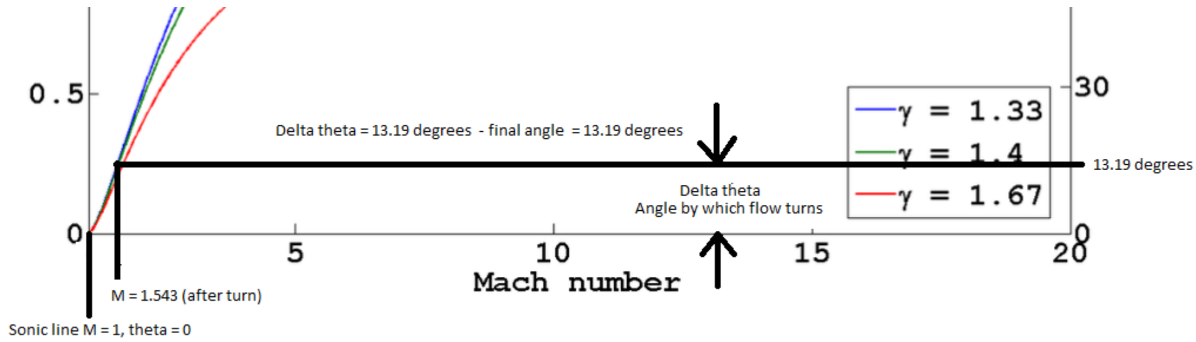
Besides this flowline, we can pick as many others as necessary starting from the same point, but at lesser angles. This is in order to generate a series of characteristic-lines which can be used in the design. In order to make this worked example simple, only one other has been chosen – at half the maximum angle (6.959°). These two flowlines are shown below:



We'll label the point at the wall (point 0 in the first-page diagram) as (0,1) in Cartesian co-ordinates. We are only interested in the C^- lines from these as the C^+ lines don't exist (they go into the wall).

Let's calculate some figures:

Let's start with the 13.19° flowline. What we are saying is actually that the flow here <u>turns</u> through 13.19°. This is because it starts (at the sonic-line) at $\theta = 0°$ and M = 1 and ends up at $\theta = 13.19°$ and M = ? so $\Delta\theta = \nu(M_2) - \nu(M_1)$. This can be represented on the PM graph for the 13.19° situation as shown:



So, from the graph, after the turn M = 1.543. The other figures are:

$$\mu = sin^{-1}\left(\frac{1}{M}\right) = sin^{-1}\left(\frac{1}{1.543}\right) = 40.38^o$$

$$K_- = \theta + \upsilon(M) = 13.19 + 13.19 = 26.38$$

$$K_+ = 0 \text{ (It's going right into the wall)}$$

$$\left(\frac{dy}{dx}\right) \text{ are not required at the moment.}$$

Calculating these for the other line and summarising:

Point	x	Y	θ	ν	M	μ	$C^+\frac{dy}{dx}$	$C^-\frac{dy}{dx}$	K^+	K^-
0	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38

At point 1

Since point 1 is on the centre-line $\theta_1 = 0$ (the flow all along the centre line is parallel to it as you might imagine from common sense).

We already know the value of K_- for the line going from point 0 to point 1 - we calculated it in the previous step as 13.19 (this is the value from the $\theta_0 = 6.595$ and $\mu_0 = 49.48°$ line). We can now use this to calculate the missing parameter at point 1 ($\nu_1(M)$). So, with the new values:

$$K_- = \theta_1 + \upsilon_1(M) = 13.19 \therefore \nu_1(M) = 13.19$$

Which gives us from the graph $M_1 = 1.543$ and $\mu_1 = 40.38$.

But where is point 1 on the x axis? We can find this from:

$$\left(\frac{dy}{dx}\right)_{C^-} = \tan(\theta - \mu)$$

Let's make two assumptions – firstly that this is a straight line so the gradient is y_2-y_1/x_2-x_1 and that the two angles (μ and θ) are the average values between point 0 and point 1. The equation then becomes:

$$\frac{y_1 - y_0}{x_1 - x_0} = \tan\left(\theta_{average} - \mu_{average}\right)$$

Filling in the known numbers we have:

$$\frac{0 - 1}{x_1 - 0} = \tan\left(\frac{6.595 + 0}{2} - \frac{49.48 + 40.38}{2}\right)$$

Work this through and you should get $x_1 = 1.125$.

Let's fill in the next row of our table (the other values should be obvious):

Point	x	Y	θ	ν	M	μ	$C^+ \frac{dy}{dx}$	$C^- \frac{dy}{dx}$	K^+	K^-
0,1	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0,2	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38
1	1.125	0	0	13.19	1.543	40.38	-	-0.888	-13.19	13.19

At point 2

Point 2 is at the intersection of two known lines:

The K_- line from point 0 (originating at $\theta = 13.19$, $\mu = 40.38$) this has a value of 26.38.

The K_+ line from point 1 which we calculated in the last step, this has a value of -13.19.

This allows us to get two simultaneous equations with two unknowns and solve for θ_2 and $\nu_2(M)$:

$$\theta_2 - \nu_2(M) = -13.19$$

$$\theta_2 + \nu_2(M) = 26.38$$

If you solve these and use the graph you should find:

$$\theta_2 = 6.595^o, \nu_2(M) = 19.785, M_2 = 1.7675, \mu_2 = 34.45^o$$

The coordinates for be found by solving for the intersect of the two lines from 0 and 1 (two simultaneous equations) as before:

$$\frac{y_2 - 0}{x_2 - 1.125} = \tan\left(\frac{0 + 6.595}{2} + \frac{40.38 + 34.45}{2}\right)$$

$$\frac{y_2 - 1}{x_2 - 0} = \tan\left(\frac{13.19 + 6.595}{2} - \frac{40.38 + 34.45}{2}\right)$$

Important note: At this point in the video the author has made an error: he has put in 1.4262 instead of 1.125 (as above). I'm going to keep going with his figures so that they align with the video – but bear in mind from now on they are incorrect (but the method is right – it's just an arithmetical error).

So, by working though his (flawed) equations the following (incorrect) values are arrived at:

$$x_2 = 1.425, y_2 = 0.2575$$

The summary table is shown:

Point	x	Y	θ	ν	M	μ	$C^+ \frac{dy}{dx}$	$C^- \frac{dy}{dx}$	K^+	K^-
0,1	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0,2	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38
1	1.125	0	0	13.19	1.543	40.38	-	-0.888	-13.19	13.19
2	1.425	0.2575	6.595	19.785	1.7675	34.45	0.8605	-0.521	-13.19	26.38

<u>At point W_1</u>

Because all we are really interested in is the x and y coordinates of point W_1 only these are calculated. The trick at the wall is that the angle of the wall should be equal to the angle of flow as this cancels out the shockwaves. The angle for flow (coming from the line generated by point 2) is 6.595°. This gives us the equation (using the 1.425 incorrect value):

$$\frac{y_{W1} - 0.2575}{x_{W1} - 1.425} = \tan(13.19 + 34.45)$$

But there's two unknowns – we get over this problem by using a straight line approximation from point 0 to point W_1:

$$\frac{y_{W1} - 1}{x_{W1} - 0} = \tan\left(\frac{13.19 + 6.595}{2}\right)$$

Solving these two we get:

$$x_{W_1} = 2.8482, y_{W_2} = 1.4967$$

Table:

Point	x	Y	θ	ν	M	μ	$C^+\frac{dy}{dx}$	$C^-\frac{dy}{dx}$	K^+	K^-
0,1	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0,2	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38
1	1.125	0	0	13.19	1.543	40.38	-	-0.888	-13.19	13.19
2	1.425	0.2575	6.595	19.785	1.7675	34.45	0.8605	-0.521	-13.19	26.38
W_1	2.8482	1.4967	6.595	-	-	-	0.8707	0.1744	-	-

<u>At point 3</u>

We take the C_- line from point 2 and this reaches the midline at point 3. This is a similar idea to the way we calculated point 1 - the K_- constant in this case is 26.38. We get:

$$\theta_3 = 0, \nu_3(M) = 26.38, M_3 = 2, \mu_3 = 30$$

And the coordinates (again using 1.425):

$$\frac{0 - 0.2575}{x_3 - 1.425} = \tan\left(\frac{6.595 + 0}{2} - \frac{34.45 + 30}{2}\right)$$

Gives us:

$$x_3 = 1.891$$

Adding into the table:

Point	x	Y	θ	ν	M	μ	$C^+\frac{dy}{dx}$	$C^-\frac{dy}{dx}$	K^+	K^-
0,1	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0,2	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38
1	1.125	0	0	13.19	1.543	40.38	-	-0.888	-13.19	13.19
2	1.425	0.2575	6.595	19.785	1.7675	34.45	0.8605	-0.521	-13.19	26.38
W_1	2.8482	1.4967	6.595	19.785	1.768	34.45	0.8707	0.1744	-13.19	26.38
3	1.891	0	0	26.38	2.0	30.00	-	-0.5527	-26.38	26.38

At point W_2

Now actually we don't have to calculate the wall angle at W_2 - because it's zero (the nozzle has completely straightened at this point), this also means $\theta_3 = 0$. Using the same reasoning as for W_1 we have, for the coordinates:

$$\frac{y_{W2} - 0}{x_{W1} - 1.891} = \tan(0 + 30)$$

$$\frac{y_{W2} - 1.4967}{x_{W2} - 2.8482} = \tan\left(\frac{6.595 + 0}{2}\right)$$

Giving:

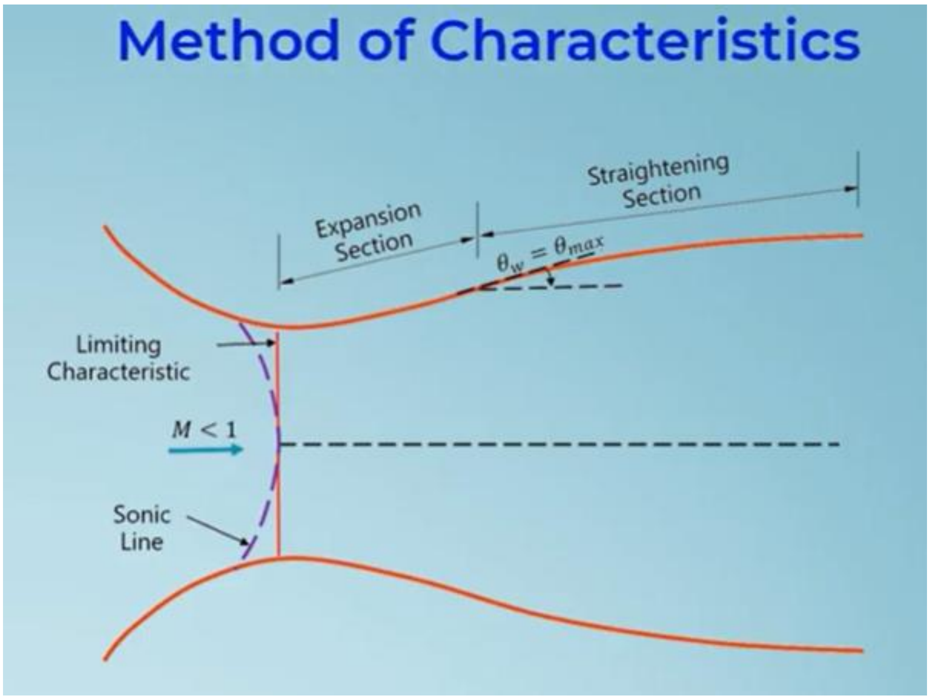
$$x_{W_2} = 4.665, y_{W_2} = 1.602$$

Table:

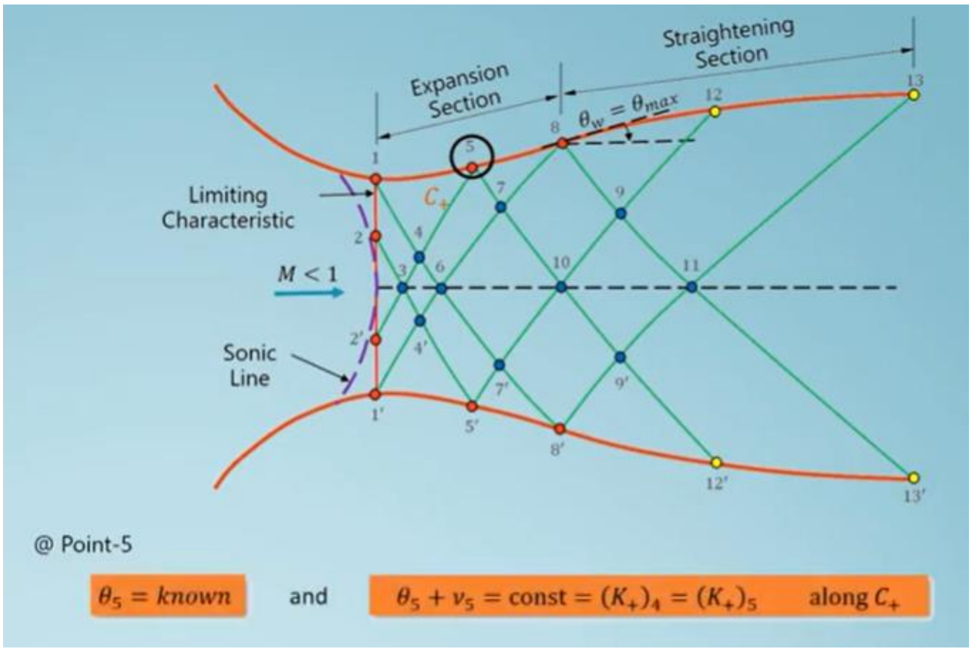
Point	x	Y	θ	ν	M	μ	$C^+ \frac{dy}{dx}$	$C^- \frac{dy}{dx}$	K^+	K^-
0,1	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0,2	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38
1	1.125	0	0	13.19	1.543	40.38	-	-0.888	-13.19	13.19
2	1.425	0.2575	6.595	19.785	1.7675	34.45	0.8605	-0.521	-13.19	26.38
W_1	2.8482	1.4967	6.595	19.785	1.768	34.45	0.8707	0.1744	-13.19	26.38
3	1.891	0	0	26.38	2.0	30.00	-	-0.5527	-26.38	26.38
W_2	4.665	1.602	0	-	2.0	-	0.5774	0.0576	-	-

Nozzle with expansion section (not a minimum length nozzle).

This situation is shown below:



Note that the maximum wall angle here is not at the throat, but at the start of the straightening section. Wall points in the expansion section have a outgoing characteristic line associated with them (unlike the straightening section where they cancel out) which can be calculated as shown:



This example can be found at: [Method of Characteristics — Lesson 6 - YouTube]